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ON SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

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1. The geometric theory of singular solutions, as it may be called, was given by Cayley in 1872, in the *Messenger of Mathematics*, Vol. II. New Series, pp. 6–12; and further illustrations were given by Cayley in Vol. VI. pp. 23–27, and by Glaisher in Vol. XI. pp. 1–14. The writer employed very much the same method in 1877, in Vol. IV of the *Analyst*, being then unacquainted with Cayley's work. The main points of the theory are as follows: The differential equation being a rational integral equation of the m^{th} degree with respect to $p\,(= dy\,/\,dx)$, the coefficients being one-valued functions of x and y, the integral is in like manner a rational integral equation of the same degree with respect to an arbitrary constant c, having coefficients of the same character. A singular solution satisfies the condition for equal roots in the p-equation; it also satisfies the same condition in the c-equation and represents the envelope of the system of curves represented by that equation. But each of these conditions is also satisfied by other relations between x and y. Namely, if $E = 0$ is the equation of the envelope of the system of curves, then the first condition, or equation in which the p-discriminant is equated to zero, is

$$ECT^2 = 0,$$

where $C = 0$ is the locus of the cusps of the system of curves, and $T = 0$ the *tac-locus*, or locus of points where two curves of the system touch one another. Again, the second condition, in which the c-discriminant is equated to zero, is

$$EC^3N^2 = 0,$$

where E and C have the same meanings as before, and $N = 0$ is the locus of the nodes, or ordinary double-points of the system of curves.

2. This last conclusion Cayley derives from a consideration of the c-discriminant as the locus of the intersections of consecutive curves of the system repre-

sented by the c-equation, showing that, if the variable curve has a permanent node or cusp, there will be two of these intersections which ultimately coincide with each node, and three which coincide with each cusp. If the curve is algebraic and of the m^{th} degree there will remain

$$m^2 - 2\delta - 3\varkappa$$

(which by Plücker's equations is $m + n$, and thus always a positive integer) intersections whose locus is the envelope. There is thus always an envelope real or imaginary, if the integral is algebraic; but as there is not generally a singular solution, it is inferred that a differential equation has not generally an algebraic integral.

3. A consideration of the subject more especially from the point of view of the differential equation, may not be without interest.

The differential equation is a relation between x, y, and p, where if x and y are the rectangular co-ordinates of a point, p is the tangent of the inclination of its motion to the axis of x. We may give to x and y any values we please, and determine m values of p. Thus the equation may be regarded as satisfied by a point having any position, provided only it be moving in one of the directions proper to its position. Consider now a moving point starting from the position (x_0, y_0) and satisfying the differential equation. As it moves, the values of x and y vary, and this in general causes p, as determined by the equation, to vary; so that the point, in order to continue to satisfy the equation, must in general move in a curve. The point may return to its original position describing a closed curve, or it may pass to an infinite distance describing an infinite branch. We have thus a curve such that, if a point move along it in either direction, it satisfies the differential equation; and if we can determine the equation of this curve we shall have a solution of the differential equation. Furthermore, if the solution so determined contains an arbitrary quantity c independent of x and y, the initial point (x_0, y_0) may be any point we please; but to each point (x, y) there must correspond as many values of c as of p, so that the initial value of p may be any one of the m values which are consistent with the values assumed for x and y. If this be the case, we shall have a complete solution of the differential equation in the sense that to every moving point representing simultaneous values of x, y, and p which satisfy the differential equation, there corresponds, if we properly determine c, a curve in which the point may be assumed to be moving. Such a solution is the complete integral.

But, if in the complete solution we are to include the equations of all the curves in which, if the point be moving, it will satisfy the differential equations, then we have still to inquire whether there be any other curves in which the point may be moving. If there be, it is clear that they must at each point coincide in direction with one of the curves included in the complete integral ; in other words, they must be envelopes of the system of curves represented by the complete integral.

4. Since, in general, there must be, corresponding to any point (x, y), as many values of c as of p, the degree of the c-equation, when reduced to its standard form, must be the same as that of the p-equation; but with regard to some special points, the following considerations may be noticed:—

When the equations are of the first degree, the standard forms are for the p-equation

$$Lp + M = 0,$$

and for the c-equation,

$$Pc + Q = 0,$$

where L, M in the first, and P, Q in the second, are one-valued functions of x and y. $L = 0$ is a locus along which p is infinite, that is a locus of points at which the particular integral curves are parallel to the axis of x; but if $x + a$ is a factor of L, $x + a = 0$ is itself a solution. $M = 0$, in like manner, is a locus along which $p = 0$; and if $y + b$ is a factor of M, $y + b = 0$ is a solution. But at the intersections of $L = 0$ and $M = 0$, p is indeterminate. The system of curves is in many cases a pencil of curves passing through these points (and having no other intersections with one another), c being indeterminate at the same points, so that $P = 0$ and $Q = 0$ intersect in these points. But in other cases, the indeterminate value of p indicates that a particular curve of the system has a node at the point in question. As simple illustrations of the two cases, take the equations $xp - y = 0$ and $xp + y = 0$; in both cases the loci in question, $x = 0$ and $y = 0$ are themselves solutions and their intersection is the origin. In the first case the integral is $x - cy = 0$, representing a pencil of straight lines through the origin; in the second the integral is $xy - c = 0$, and the special solutions $x = 0$ and $y = 0$ make up the particular integral corresponding to $c = 0$, namely $xy = 0$, which has a node at the origin.

5. When the equations are of the second degree, the standard form of the p-equation is

$$Lp^2 + Mp + N = 0$$

where L, M, and N are one-valued functions of x and y. There are generally two values of p at the point (x, y). $L = 0$ is a locus along which one of the value of p is infinite; $N = 0$ is a locus along which one of the values of p is zero There are generally no points at which p is indeterminate, since at the intersection of $L = 0$ with $N = 0$ we have not generally $M = 0$. The locus of points for which the two values of p are equal, is

$$M^2 - 4LN = 0.$$

The first member of this equation, or p-discriminant, is the square of the differences of the roots of the p-equation. If these roots are rational functions of and y, the discriminant is the square of a rational function; the equation can then be decomposed into two equations of the first degree, which give rise to two distinct systems of curves. Through every point (x, y) passes one curve of each system corresponding to the two values of p; but when (x, y) is on the discriminant locus, there is but one value of p, and the two curves, one of each system, touch one another. There is, in this case, no propriety in combining the two integrals into a single expression as Boole does, because there is no determinate manner of pairing together, so to speak, the curves of one system with those of the other, as is done in the assumption that the constant of integration is the same in each system.

6. In the proper, or indecomposable equation of the second degree, the two values of p are the two values of an irrational function of x and y, their difference is a real quantity when the roots are real, and a purely imaginary quantity when the roots are imaginary; thus the discriminant is positive when p has real values, and negative when the values of p are imaginary. The discriminant may break up into factors, and we have to consider the loci on which each of these factors vanish. We have particularly to distinguish between those which occur in the discriminant with an even exponent, so that when (x, y) passes over the locus the discriminant does not change sign, and the values of p remain real; and those which occur with an odd exponent, so that the value of p becomes imaginary as (x, y) passes over the locus.

There is no reason whatever to expect that the values of p when they become equal, should be identical with the value of p for a point moving along the locus of equal roots; that is, that the locus itself should be a solution of the differential equation. This may, however, happen and will constitute a special case to be examined.

7. Imagine now the point (x, y) carrying with it the two intersecting arcs of particular integrals which pass through it, to be moved up to one of the loci of equal roots; and first, in the general case, where the factor which vanishes at the locus occurs in the discriminant in the first (or any odd) degree, and the value of p at the locus is not that corresponding to a point moving along the locus. When the point (x, y) is near the locus, the two arcs passing through it meet the locus at an angle, and do not cross it; and when the point is moved up to the locus they become tangent to one another and form a cusp The two arcs now belong to the same curve, and there is corresponding to the point (x, y), when on the locus, a single value of c. Hence the factor which vanishes at the locus appears also in the c-discriminant; and, because c becomes imaginary when p does, it therein appears with an odd exponent. In fact, as before mentioned, it appears as a cubic factor; the possession of a cusp is of course a peculiarity of the integral equation, implying the existence of a relation between

the coefficients, but it is not such with respect to the differential equation, just as it is not with respect to the tangential equation of a curve.

In the next place, still supposing the factor which indicates the locus to be of an odd degree, suppose the value of p to be the same as that for a point moving along the locus. In this case, when (x, y) is moved up to the locus the two arcs do not, indeed, cross the locus, but they become tangent to it, and each forms the ordinary continuation of the other. There is, as before, a single value of c corresponding to the point (x, y) when on the locus, and the factor enters the c-discriminant with an odd exponent, generally unity. The locus is an envelope, which, unlike the cusp-locus, does not imply any peculiarity in the integral equation.

In the third place, suppose the factor which vanishes at the locus to be a squared factor, so that the value of p does not become imaginary as we cross the locus, and that the equal values of p differ from the value of p for a point moving in the locus. In this case, the two arcs cross the locus, and when (x, y) is moved up to the locus they simply become tangent to one another, the values of c remaining distinct; we have the tac-locus, the factor not appearing at all in the c-discriminant.

Finally, the locus being still represented by a squared factor, suppose the equal values of p to be identical with the value of p for a point moving in the locus. In this case, the two arcs of particular integrals, instead of crossing the locus, when (x, y) is moved up to it, will coincide with it in direction as in the case of the envelope. But since p is real on both sides of the locus, when (x, y) is moved across the locus, the arcs will reappear on the other side, passing, as it were, bodily across the locus. This implies that the arcs coincide with the locus when (x, y) is upon it. Thus the locus coincides with a particular integral;* namely, an integral which is given doubly in the system, either by a single value of c, or as parts of the integrals given by different values of c. In the first of these cases, the factor will appear in the c-discriminant with an even exponent; in the second case the factor will not appear in that discriminant at all.

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*This conclusion, or the equivalent statement that an envelope cannot be represented in the p-discriminant by a squared factor, requires modification when the degree of the p-equation is higher than the second, from the fact that more than two values of p can become equal. To show this, imagine a curve with a point of inflexion at A; and let B and C be neighboring points, one on either side of A, at which the tangents are parallel. Let the system of curves be the result of moving this curve in the direction of the tangents at B and C. These tangents will then be envelopes; each of them separates a region of the plane in which p has at every point three real values, from one in which p has but one real value, the p-equation being, suppose, of the third degree. Now suppose the points B and C to be taken nearer to A, the region in which the three roots are real is narrowed, the two envelopes approaching, until they finally coincide. The envelope is now represented by a squared factor; only one value of p is real on either side of it, but as we cross it the three roots become equal at once. Thus, when $n > 2$, a squared factor of the p-discriminant may represent an envelope which touches the curves of the system at points of inflexion and at which three values of p become equal. Similar conclusions follow when more than three value of p become equal.

8. The relation between the several cases enumerated above is well illustrated by means of the orthogonal trajectories of the given system of curves. The differential equation of the orthogonal trajectories, being the result of substituting $-1/p$ for p in the given differential equation, has precisely the same discriminant as the given equation. Now the trajectories will meet an envelope of the given system at right angles; and since the values of p become imaginary in both equations as we cross the locus, the envelope will be a cusp-locus of the trajectories. Conversely, a cusp-locus which is at each point perpendicular to the curves of the given system, becomes an envelope of the trajectories; but every other cusp-locus is also a cusp-locus of the trajectories. In like manner, a tac-locus becomes a tac-locus of the trajectories, except when it crosses the touching curves of the system at right angles, in which case it is itself a trajectory, and comes under the fourth of the cases considered above.

9. The consideration of the orthogonal trajectories serves also to illustrate the relation between the cases in which p is at a certain point indeterminate; for such points evidently have the same character in each of two mutually orthogonal systems. Thus, resuming the illustrations given in § 4, $xp - y = 0$ gave us a pencil of lines passing through the origin; the differential equation of the trajectories is $x + py = 0$, which gives $x^2 + y^2 = c^2$, a system of circles in which the origin is an acnode corresponding to $c = 0$. In the other case, $xp + y = 0$, the system represented was the system of hyperbolas $xy - c = 0$, where the origin was a crunode corresponding to $c = 0$; and here the differential equation of the trajectories is $x - py = 0$, which gives $x^2 - y^2 = c^2$, another system of hyperbolas in which the origin is also a crunode.